Let $(W(k), k \in \mathbb{T})$ be the process formed from the increments of the fractional Brownian motion as

$$W(k) = B^H(k+1) - B^H(k) \tag{24}$$

for $k \in \mathbb{T}$. The conditional expectations for U^* in (4) are obtained by predicting the future increments of the fractional Brownian motion given the past increments. The required computations for these predictions are known for general Gaussian processes. To apply these results to the sequence formed by the increments of a fractional Brownian motion let $k > \ell$. By a Gram-Schmidt orthogonalization procedure there is a collection of independent standard Gaussian random variables, $\{V(0), \dots, V(\ell)\}$, that is equivalent to $\{W(0), \dots, W(\ell)\}$. Then

$$\begin{aligned} &\mathbb{E}\left[W(k)|W(0), \dots, W(\ell)\right] \\ &\quad = \mathbb{E}\left[W(k)|V(0), \dots, V(\ell)\right] \\ &\quad = \Sigma_{j=1}^{\ell} \mathbb{E}\left[W^T(k)V(j)\right] V(j) \qquad \text{a.s.} \end{aligned} \tag{25}$$

The computations of the expectations follow from (23).

V. Conclusion

The completion of squares method that is used here allows for a system with a general square integrable noise process. The (7) that is used here is somewhat analogous to the equation for the continuous time results [2]–[4]. However many discrete time linear equations provide the same limiting continuous time equations, so an elementary discretization of the continuous time equations does not yield the discrete time results. It should also be important to solve the associated partially observed discrete time control problem.

References

[1] M. Aoki, *Optimization of Stochastic Systems*. New York: Academic Press, 1967.

[2] T. E. Duncan and B. Pasik-Duncan, "Stochastic linear-quadratic control for systems with a fractional Brownian motion," in *Proc. 49th IEEE Conf. Decision Control*, Atlanta, GA, 2010, pp. 6163–6168.

[3] T. E. Duncan and B. Pasik-Duncan, "Linear quadratic fractional Gaussian control," Preprint.

[4] T. E. Duncan, B. Maslowski, and B. Pasik-Duncan, "Linear-quadratic control for stochastic equations in a Hilbert space with fractional Brownian motions," *SIAM J. Control Optim.*, vol. 50, pp. 507–531, 2012.

[5] P. D. Joseph and T. T. Tou, "On linear control theory," *Trans. AIEE (Appl. Ind.)*, vol. 80, pp. 193–196, 1961.

[6] R. E. Kalman and R. W. Koepcke, "Optimal synthesis of linear sampling control systems using generalized performance indices," *Trans. ASME*, vol. 80, pp. 1820–1826, 1958.

[7] J. B. Moore, X. Y. Zhou, and A. E. B. Lim, "Discrete time LQG controls with control dependent noise," *Syst. Control Lett.*, vol. 36, pp. 199–206, 1999.

Adaptive Tracking Control of Linear Systems With Binary-Valued Observations and Periodic Target

Yanlong Zhao, Jin Guo, and Ji-Feng Zhang

***Abstract*—This technical note studies the adaptive control for linear systems with set-valued observations to track a given periodic target. Based on the system parameters, accessorial parameters with the same order as that of the tracking targets are introduced and estimated. Considering the system parameters are unknown and set-valued observations can supply only limited information each time, a two-scale adaptive control algorithm is designed. Each control input is designed at the large time scale and lasts for a holding time (small scale), during which the parameter estimation algorithm is constructed. From the estimate of accessorial parameters, the control signal is updated at the large time scale by the certainty equivalence principle. As the holding time goes to infinity, the algorithm can be proved to be asymptotically efficient in a certain sense. Meanwhile, the adaptive tracking algorithm is shown to be asymptotically optimal. A numerical example is given to demonstrate the effectiveness of the algorithms and the main results obtained.**

***Index Terms*—Adaptive control, asymptotically efficient, asymptotically optimal control, parameter estimation, periodic target.**

I. Introduction

The theory on system control and identification has played important roles in areas of social, financial, biological, industrial, and medical systems. With the solving of practical problems, the theory itself has been developed and generated more and more catalogues, from continuous ones to discrete ones, from time-invariant ones to time varying ones, from single systems to multi-agent systems, and so on. Many classic methodologies have been raised such as least-squares algorithm ([1], [2]), maximum likelihood algorithm ([3], [4]), Kalman filtering ([2], [5]), self-tuning regulator ([6]). These methodologies are constructed based on the accurate information of system output, or output with measurement noises.

With the development of new technologies, set-valued output systems have appeared, which can only supply whether the system output is in some set or not, to challenge the base of classic identification and control methods. For example, the nerve cell ([7]) has its action potential dynamics and a fixed threshold inside. If the action potential is larger than the threshold, then the cell is in the state of excitation; otherwise, inhibition. From the outside of the cell, we can only tell whether the cell is excited or inhibited, or equivalently, whether the action potential is in the set of less than the threshold. The nerve cell is an example of set-valued information with fixed thresholds, the case with the adjustable thresholds can be found in the so called quantized observations in wireless sensor networks ([8], [9]).

In both cases, the information of set-valued observations is much more limited than the one of accurate observations. The measured

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signals to the input, state and controlled output are not one to one mappings, but essentially nonlinear. Identification and adaptive control methods for conventional systems cannot be applied to such systems. Thus, new algorithms and theory are needed to be developed for parameter identification, adaptive control and performance analysis of the set-valued observation systems.

On parameter identification, state estimation and stabilization control of such systems, there have already been some initial works ([9]–[14]). On parameter identification, [12] and [13] gave a strongly consistent and asymptotically optimal algorithm based on the periodic input and statistical properties of the system noises. [10] discussed the linear system identification with the colored noises based on multi-sine input signal. [14] studied the identification of quantization systems under a class of deterministic persistent excitation inputs. On stabilization control, [11] considered the case where the parameters were known, and proposed a state observer and a stabilization control.

On adaptive control of systems with set-valued observations, few works appear in literature. The main reason is that for identification purpose the system input can be assumed to be periodic, persistently excited, or normally distributed; while for feedback control, the input signal is decided by the control target, which spoils the above assumptions on inputs. However, [15] studied the adaptive tracking control of a class of one-order systems with binary valued observations and time-varying thresholds. What is different from [15] is that this technical note discusses the high order systems with fixed thresholds. The fixed thresholds are defined in advance and the time-varying ones can be designed according to the actual need, so the case of fixed thresholds supplies less information than the one of time-varying ones. On the other hand, the increase of parameters makes the coupling of the states stronger. As a result, the method in [15] does not work here.

In this technical note, we try to challenge the adaptive control problem with binary-valued observations, which is the base of set-valued observations, for FIR (Finite Impulse Response) model with independent and identically distributed (i.i.d.) noises. The tracking target is periodic and the control is designed under mean square performance. Since the system parameters are unknown and set-valued observations can supply only limited information each time, we construct a two-scale adaptive control algorithm: the scale for identification is small and the scale for control is large. For each control input, it will last for a holding time, during which the system parameter is estimated, and the estimate is updated at the end of the holding time. Based on the estimate of the parameters and the target of the system output, the control law is designed. As the holding time increases to infinity, the adaptive tracking is shown to be asymptotically optimal.

This technical note is organized as follows. Section II formulates the problem and gives the tracking performance. Then, the two-scale adaptive algorithm is constructed in Section III. Consequently, the properties of both the identification and adaptive control are analyzed in Section IV. Section V uses a numerical example to demonstrate the effectiveness of the algorithms and the main results obtained. Section VI introduces the potentially extended works that can be studied based on the results in this technical note. Finally, Section VII gives some concluding remarks.

II. Problem Formulation

Consider the FIR system

$$y(k) = \phi^T(k)\eta + d(k), \quad k = 1, 2, \dots \tag{1}$$

where $\phi(k) = [u(k-1), u(k-2), \dots, u(k-n)]^T$ is the vector of the input with $u(k) = 0$ for $k < 0$, $\eta = [a_1, \dots, a_n]^T$ is a vector of unknown but constant parameters and $d(k)$ is the system noise.

The system output $y(k)$ is measured by a binary-valued sensor with the threshold $C \in (-\infty, \infty)$, which can be represented by an indicator function

$$s(k) = I_{\{y(k)\le C\}} = \begin{cases} 1, & \text{if } y(k) \le C; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Assumption 2.1: $\{d(k), k = 1, 2, \dots\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with zero mean and known covariance σ^2. The distribution function of $d(1)$, denoted by $F(\cdot)$, is known and twice continuously differentiable.

Assumption 2.2: The *prior* information on the unknown parameter η is that $\eta \in \Omega \subseteq \mathbb{R}^n$ and Ω is a known compact set with $\sup_{\omega\in\Omega} \|\omega\|_2 \triangleq M < \infty$, where $\mathbb{R}^n$ means the n dimensional space of real numbers and $\|\cdot\|_2$ is the Euclidean norm of "$\cdot$".

Assumption 2.3: There exists a known constant $\delta \in (0, 1)$ such that $|B(z)| > \delta$ for $|z| \le 1$ with $B(z) \triangleq a_1 + a_2 z + \dots + a_n z^{n-1}$.

Remark 1: In case of $\delta = 0$, the condition in Assumption 2.3 turns to be the minimum phase condition, which is necessary for adaptive tracking problem even if the system parameters are known ([6]). In this technical note, the condition is a little stronger in order to avoid the singularity of the accessorial parameter matrix constructed in the next section.

Definition 1: ([16]): An $m \times m$ circulant matrix

$$T = \begin{bmatrix} v_m & v_{m-1} & v_{m-2} & \cdots & v_1 \\ v_1 & v_m & v_{m-1} & \ddots & v_2 \\ v_2 & v_1 & v_m & \ddots & v_3 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ v_{m-1} & v_{m-2} & v_{m-3} & \cdots & v_m \end{bmatrix} \tag{3}$$

is completely determined by its first row $[v_m, \dots, v_1]$, which will be denoted by $\mathbf{T}([v_m, \dots, v_1])$.

Definition 2: The periodic signal $v_1, \dots, v_m, v_1, \dots, v_m, \dots$ generated from $v = [v_m, \dots, v_1]^T$ is of full rank if the circulant matrix $\mathbf{T}([v_m, \cdots, v_1])$ is of full rank. And the vector v is also called full rank.

Assumption 2.4: The periodic target signal $\{y^*(k), k = 1, 2, \dots\}$ is periodic with $y^*(1) = y_1^*$ and one period $(y_1^*, \dots, y_m^*)$, which is known and deterministic. And, $\mathbf{y}^* \triangleq [y_m^*, \dots, y_1^*]^T$ is of full rank.

Remark 2: Assumption 2.4 describes the properties of the reference signals, based on which a control law can be designed to ensure a sufficient persistent excitation condition for parameter estimation.

The goal of this technical note is to design a control law to make the system output $y(k)$ track the given periodic signal $y^*(k)$ based on the binary-valued observation $s(k)$ in the case of unknown system parameters and minimize the index

$$J(k) = E\left[\sum_{j=1}^{m} \left(y(k+j) - y_j^*\right)^2\right] \tag{4}$$

for $k = ml, l = 1, 2, \dots$.

III. Design of Adaptive Control Law

This section will provide a two-scale method of constructing the adaptive control law. For convenience, we first transform the problem to another form.

A. Transformation of the Problem

To clarify our thinking, we consider the case where the parameters are known. By (1) and Assumption 2.1, we have $E\sum_{j=1}^{m}\left(y(k+j) - y_j^*\right)^2 =$

$E\sum_{j=1}^{m}\left(\sum_{i=1}^{n}u(k+j-i)a_i-y_j^*+d(k+j)\right)^2 = mEd^2(1)+\sum_{j=1}^{m}\left(\sum_{i=1}^{n}u(k+j-i)a_i-y_j^*\right)^2$, which implies that the control law of minimizing (4) should satisfy

$$\sum_{i=1}^{n}u(k+j-i)a_i=y_j^*,\quad j=1,\ldots,m.$$

Thus, $\min_{k\geq 1}J(k)=mEd^2(1)=m\sigma^2$ and the optimal control law is an m-periodic signal.

Inspired by this, we represent the system (1) as another form where the dimension of the input is m. To do so, the system output, input and noise are written in the vector form as

$$\begin{aligned}\mathbf{y}(l)&=[y(lm),\ldots,y((l-1)m+1)]^T\in\mathbb{R}^m,\\ \mathbf{s}(l)&=[s(lm),\ldots,s((l-1)m+1)]^T\in\mathbb{R}^m\\ \Phi(l)&=[\phi(lm),\ldots,\phi((l-1)m+1)]^T\in\mathbb{R}^{m\times m},\\ D(l)&=[d(lm),\ldots,d((l-1)m+1)]^T\in\mathbb{R}^m\end{aligned}$$

for $l=1,2,\ldots$. Accordingly, the tracking index can also be updated as

$$J(l)=E\left(\mathbf{y}(l)-\mathbf{y}^*\right)^T\left(\mathbf{y}(l)-\mathbf{y}^*\right).\tag{5}$$

Noticing that the matrix of the input $\Phi(l)$ is an $m\times m$ one, the first problem is to construct an accessorial parameter $\theta=[b_1,\ldots,b_m]^T$ with m dimension such that the system output can be rewritten into a vector form as

$$\mathbf{y}(l)=\Phi(l)\theta+D(l).\tag{6}$$

Now, we show how to generate b_i, $i=1,\ldots,m$. Considering the comparison of n and m, there exist three situations:

i) As $n=m$, let $b_i=a_i$, $i=1,\ldots,m$;
ii) As $n<m$, let $b_i=a_i$ for $i=1,\ldots,n$ and $b_i=0$ for $i=n+1,\ldots,m$;
iii) As $n>m$, let

$$b_i=\sum_{j=0}^{\left\lfloor\frac{(n-i)}{m}\right\rfloor}a_{i+jm}\tag{7}$$

for $i=1,\ldots,m$, where $\lfloor\cdot\rfloor$ means the largest integer less than or equal to "$\cdot$".

The accessorial parameter θ plays a very important role during the control design in this technical note. An advantage of the above construction method is shown by the following proposition, whose proof can be found in Appendix A.

Proposition 1: The accessorial parameter matrix $\mathbf{T}([b_m,\ldots,b_1])$ is of full rank if $\eta=[a_1,\ldots,a_n]^T$ satisfies Assumption 2.3.

Till now, the transformation from (1) to (6) has been completed and the problem can also be restated: the purpose of this technical note is to design an adaptive control to drive the controlled output $\mathbf{y}(l)$ for the system (6) to follow a known reference signal $\mathbf{y}^*$ and minimize the tracking index $J(l)$ given by (5).

B. Adaptive Control Law

If the system parameter is known, the optimal control input matrix Φ should be designed as

$$\Phi=Y\Theta^{-1}\tag{8}$$

with $Y=\mathbf{T}([y_m^*,\ldots,y_1^*])$ and $\Theta=\mathbf{T}([b_1,b_m,\ldots,b_{m-1}])$. And also, we can conclude that Φ is of full rank based on Assumption 2.4.

In the case of unknown parameters, a two-scale adaptive control algorithm will be given. The control scale is large, each control signal lasts for a holding time, during which the small time scale is set, the system parameters are estimated, and the estimates are updated at the end of the holding time. Based on the estimates of the parameters and the tracking target, the control law is designed via the certainty equivalence principle.

Denote

$$\begin{aligned}g(t)&=\frac{t(t+1)}{2},\\ x|_\varepsilon&=xI_{\{\varepsilon\leq x\leq 1-\varepsilon\}}.\end{aligned}$$

By the analysis of Proposition 1, we have $\det(\Theta)\neq 0$. This implies that there exists $\varepsilon_0>0$, for example, $\varepsilon_0=\delta^n$ where δ is given by Assumption 2.3, such that $|\det(\Theta)|\geq\varepsilon_0$. With this information, denote

$$\varepsilon_1=\varepsilon_0/2\quad\text{and}\quad\varepsilon_2=\left(1-F\left(nC+M\|Y\|_2/\varepsilon_1\right)\right)/2$$

where $F(\cdot)$ and M are given by Assumptions 2.1–2.2 and C is the threshold in (2).

The whole control strategy is constructed as follows:

Step 0: Initial conditions: Let $\Phi(1)=I_m$ and $\widehat{\Theta}(0)=I_m$, where I_m is an m-dimension identity matrix.

Step 1: Parameter estimation based on $\Phi(g(t-1))$, $t=2,3,\ldots$.

Set $\Phi(l)=\Phi(g(t-1))$, $l=g(t-1),g(t-1)+1,\ldots,g(t)-1$, which means that $\Phi(g(t-1))$ will last for $g(t)-1-g(t-1)=t-1$ steps.

Let

$$\xi_t=\left.\left(\frac{1}{t-1}\sum_{l=g(t-1)}^{g(t)-1}\mathbf{s}(l)\right)\right|_{\varepsilon_2}\tag{9}$$

and

$$\widehat{\theta}\left(g(t)\right)=\Phi\left(g(t-1)\right)^{-1}\left(C-F^{-1}(\xi_t)\right)\mathbb{1}_m\tag{10}$$

where $\mathbb{1}_m$ is an m-dimensional column vector with each component being 1.

The estimate of the parameter matrix Θ can be constructed by $\widehat{\theta}(g(t))$. However, the matrix estimation might be singular during the calculation, so it is updated as

$$\widehat{\Theta}(g(t))=\begin{cases}\widehat{\Theta}\left(g(t)\right), & \text{if}\quad\left|\det\left(\Theta\left(g(t)\right)\right)\right|>\varepsilon_1;\\ \widehat{\Theta}\left(g(t-1)\right), & \text{otherwise}.\end{cases}\tag{11}$$

At time $l=g(t)+1,\ldots,g(t+1)-1$, let

$$\widehat{\Theta}(l)=\widehat{\Theta}\left(g(t)\right).\tag{12}$$

Step 2: Adaptive control design.

As mentioned earlier, the optimal control law should be designed by $\Phi=Y\Theta^{-1}$ in the case of known parameters. Since we have had the parameter estimation matrix $\widehat{\Theta}(g(t))$, according to the certainty equivalence principle ([6]), the adaptive control law should be designed as

$$\Phi\left(g(t)\right)=Y\widehat{\Theta}\left(g(t)\right)^{-1}.\tag{13}$$

Go back to Step 1.

Precisely speaking, the two-scale adaptive control algorithm is called since the control input is only designed at time $g(t)$, $t=2,3\ldots$, which introduces the large scale. Each control input will last a holding time from $g(t)$ to $g(t+1)-1$, which is the small scale of identification. Even the parameter estimate is only updated at time $g(t)$, $t=2,3,\ldots$, the information of the whole holding time is used

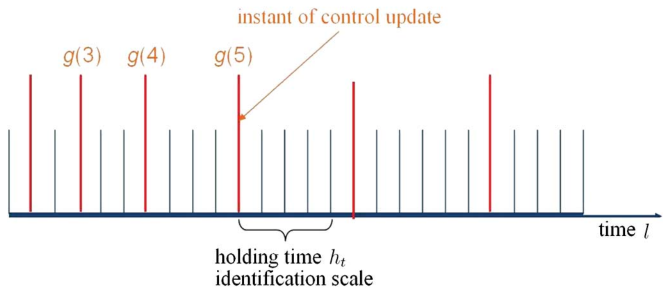

Fig. 1. Two-scale adaptive control algorithm.

by (9). At the large time scale, the control is updated based on the parameter estimate. The whole idea can be shown in Fig. 1.

Remark 3: The holding time, which is denoted as h_t, is chosen as t in this technical note. It is crucial to the two-scale algorithm and has a very important influence on the convergence speed of the parameter estimation error, which will be shown in Section IV. From the construction process of the two-scale adaptive control law, it can be seen that: i) $\Phi(l)$ is updated only at $l = \sum_{t=1}^{\tau} h_t$ for $\tau = 1, 2, \ldots$, and will last for $h_\tau = \tau$ steps after the update; ii) for a given l, $\Phi(l)$ has been updated for $N(l)$ times with

$$N(l) = \max\left\{\tau \in \mathbb{Z}_+ : \sum_{t=1}^{\tau} h_t \le l\right\} \tag{14}$$

where $\mathbb{Z}_+$ is the set of positive integers.

IV. Performance of the Closed-Loop System

In this section, it will be shown that the control law designed by Section III not only can make the adaptive tracking asymptotically optimal, but also can ensure the convergence and efficiency of the parameter estimation.

A. Parameter Estimation Properties

Denote the estimate error as $\widetilde{\theta}(l) = \widehat{\theta}(l) - \theta$, $l = 1, 2, \ldots$. Then, we have the following results:

Theorem 1: For the system (6), if Assumptions 2.1–2.4 hold, then the estimation algorithm (10) is asymptotically efficient in the sense of

$$\lim_{t\to\infty} t\left(E\widetilde{\theta}\left(g(t)\right)\widetilde{\theta}\left(g(t)\right)^T - \sigma_{CR}^2\left(g(t)\right)\right) = 0 \tag{15}$$

where $\sigma_{CR}^2(g(t))$ means the Cramér-Rao lower bound of the estimation error of θ based on $\{\mathbf{s}(l)\}$, $l = g(t-1), \ldots, g(t)-1$, and given by

$$\sigma_{CR}^2\left(g(t)\right) = \left(t\Phi\left(g(t)\right)\Lambda\left(g(t)\right)\Phi\left(g(t)\right)^T\right)^{-1} \tag{16}$$

where $\Lambda(g(t)) = \mathrm{diag}\{\lambda_1(g(t)), \ldots, \lambda_m(g(t))\}$ with

$$\lambda_i\left(g(t)\right) = \frac{f^2\left(C - \zeta_i\left(g(t)\right)\right)}{F\left(C - \zeta_i\left(g(t)\right)\right)\left(1 - F\left(C - \zeta_i\left(g(t)\right)\right)\right)}$$

$\zeta_i(g(t))$ being the i-th component of $\Phi(g(t))\theta$ for $i = 1, \ldots, m$ and $f(\cdot)$ being the density function of $d(1)$.

Proof: From the parameter estimation step in Section III-B, we have $\Phi(l) = \Phi(g(t))$, $l = g(t), g(t)+1, \ldots, g(t+1)-1$, which means that $\Phi(g(t))$ will last for $g(t+1) - 1 - g(t) = t$ steps. By Lemma 9 of [13], the Cramér–Rao lower bound of the estimation error (16) can be obtained. Then, based on Theorem 5.7 of [17], (15) is true.∎

Theorem 2: Under the condition of Theorem 1, the parameter estimates given by (10))–(13) converge to their true values with probability 1 and have the following convergence speed:

$$E\widetilde{\theta}(l)^T\widetilde{\theta}(l) = O(1/\sqrt{l}). \tag{17}$$

Proof: From (16), we have $\sigma_{CR}^2(g(t)) = O(1/t)$. In addition (15), one can get

$$E\widetilde{\theta}\left(g(t)\right)\widetilde{\theta}\left(g(t)\right)^T = O\left(\frac{1}{t}\right).$$

Notice that at any time l, the convergence speed of the parameter estimate is decided by the largest $g(t)$ less than l from (14). By the fact that $g(t) = t(t+1)/2$, the convergence speed described by (17) can be obtained. ∎

Remark 4: From Theorem 2 and the relationship between θ and η, it can be seen that η in (1) can be identified by (10)–(13) if $m \ge n$. Otherwise, only m elements of η can be done.

Remark 5: From the point view of identification, the convergence speed of the parameter estimate should be $O(1/l)$ for periodic inputs and binary observations [17]. In the adaptive tracking case, the inputs cannot be purely periodic since the system parameters are unknown and their estimates are updated. As a result, the convergence speed is slowed down to $O(1/\sqrt{l})$ by Theorem 2. In fact, the convergence speed of the parameter estimate can be faster by choosing other holding time, e.g., $h_t = \lfloor t^\lambda \rfloor$ with $\lambda > 0$, which is the largest integer no larger than t^λ. Then, for any given $\epsilon \in (0, 1)$, if the adaptive control law $\Phi(l)$ updates its value at the time $l = \sum_{t=1}^{\tau}\lfloor t^\lambda \rfloor$ for $\tau = 1, 2, \ldots$ with $\lambda = 1/\epsilon - 1$ and lasts for h_τ steps after the update, then the parameter estimate given by (10)–(13) has the speed of $O(1/h_{N(l)}) = O(1/l^{1-\epsilon})$ by the same reason as Theorem 1, where $N(l)$ is given by (14) with $h_t = \lfloor t^\lambda \rfloor$. In such sense, the two-scale adaptive control algorithm can generate the parameter estimates converging to their true values with a speed that can be arbitrarily close to $1/l$ by selecting a suitable holding time h_t. However, the convergence speed of $1/l$ cannot be achieved since the empirical estimate method (10) requires the small scale to generate periodic inputs. As a result, the two scales design is necessary for the method in this technical note.

B. Asymptotical Optimality of the Adaptive Control

Theorem 3: Under the condition of Theorem 2, the adaptive control law (10)–(13) is asymptotically optimal:

$$E\left(\mathbf{y}(l) - \mathbf{y}^*\right)^T\left(\mathbf{y}(l) - \mathbf{y}^*\right) = m\sigma^2 + O(1/\sqrt{l}). \tag{18}$$

Proof: From (6) and (13), we have

$$\begin{aligned}
&E\left(\mathbf{y}(l) - \mathbf{y}^*\right)^T\left(\mathbf{y}(l) - \mathbf{y}^*\right)\\
&\quad = E\left(\Phi(l)\theta + D(l) - \Phi(l)\widehat{\theta}(l)\right)^T\\
&\qquad \times \left(\Phi(l)\theta + D(l) - \Phi(l)\widehat{\theta}(l)\right)\\
&\quad = E\left(\Phi(l)\widetilde{\theta}(l) - D(l)\right)^T\left(\Phi(l)\widetilde{\theta}(l) - D(l)\right)\\
&\quad = E\widetilde{\theta}(l)^T\Phi(l)^T\Phi(l)\widetilde{\theta}(l) - 2ED(l)^T\Phi(l)^T\widetilde{\theta}(l)\\
&\qquad + ED(l)^TD(l).
\end{aligned} \tag{19}$$

In addition, by (10) and (13), we know that

$$\Phi(l) \in \mathcal{F}(l-1), \quad \widetilde{\theta}(l) \in \mathcal{F}(l-1) \tag{20}$$

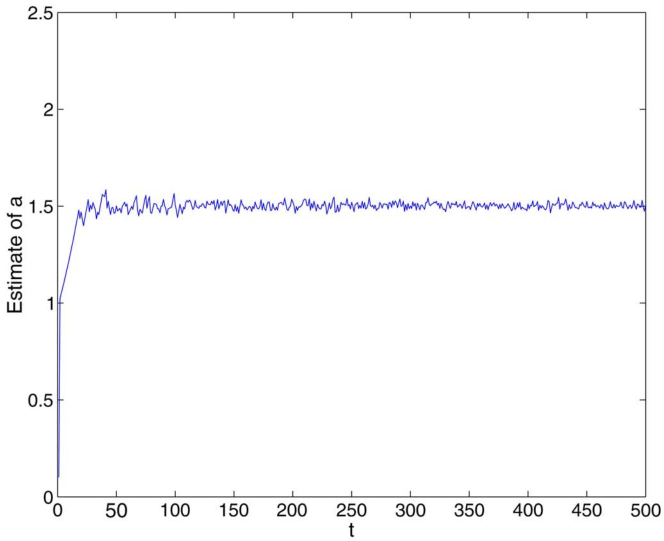

Fig. 2. Parameter estimate with real parameter $a = 1.5$ in one holding time.

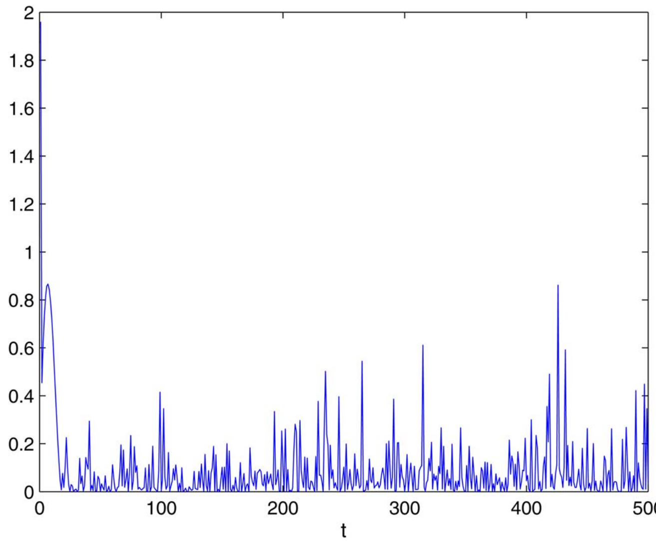

Fig. 3. Trajectory of $t(\widehat{a}(t) - a)^2$ in one holding time.

where $\mathcal{F}(l-1)$ is the σ-algebra generated by $D(1), \ldots, D(l-1)$.

By (20) and Assumption 2.1, it can be seen that

$$ED(l)^T \Phi(l)^T \widetilde{\theta}(l) = 0 \quad \text{and} \quad ED(l)^T D(l) = m\sigma^2$$

which together with (19) implies that

$$E\left(\mathbf{y}(l) - \mathbf{y}^*\right)^T \left(\mathbf{y}(l) - \mathbf{y}^*\right) = E\widetilde{\theta}(l)^T \Phi(l)^T \Phi(l) \widetilde{\theta}(l) + m\sigma^2.$$

Thus, the theorem is true by Theorem 2. ∎

Remark 6: From Theorem 2 and 3, the two-scale algorithm has achieved the goals of adaptive control and parameter convergence simultaneously. As mentioned in Remark 2, an important condition is that the target vector is of full rank, which may not be necessary if only for the adaptive tracking.

V. Simulation

Consider a gain system: $y(k) = au(k-1) + d(k)$, $k = 1, 2, \ldots$, where $a = 1.5$ is the system parameter to be identified, $d(k)$ is i.i.d. normally distributed noise with mean 0 and covariance 0.25. The output $y(k)$ is measured by a binary sensor with threshold $C = 6$, i.e., $s(k) = I_{\{y(k) \le C\}}$. The *prior* information of the system parameter is $a \in [1, 5]$. Under the binary information $s(k)$, the goal is to design control law such that the system output $y(k)$ can track the target output with $y^* = 7$.

The two-scale adaptive algorithm in Section III-B is used. The initial input of the system is set as $u(0) = 1$ and $\widehat{a}(0) = 1$, and the control $u(g(t))$ is designed by (13) at time $g(t) = t(t+1)/2$ for $t = 2, 3, \ldots$ and lasts for t steps. The parameter is estimated by (11), (12). The convergence of the identification algorithm by (10) during one holding time is shown in Fig. 2, and convergence rate can be shown by $t(\widehat{a}(t) - a)^2$ in Fig. 3, which means during one holding time the convergence speed of the estimation error covariance is $O(1/t)$.

Fig. 4 describes a trajectory of the adaptive control law, where the control signal is updated at the large scale time $k = g(t)$ and lasts for t steps for $t = 1, 2, \ldots$.

The adaptive tracking is shown in Fig. 5. The curve in Fig. 5 looks like white noise with mean 0, which shows that the system output is well tracked.

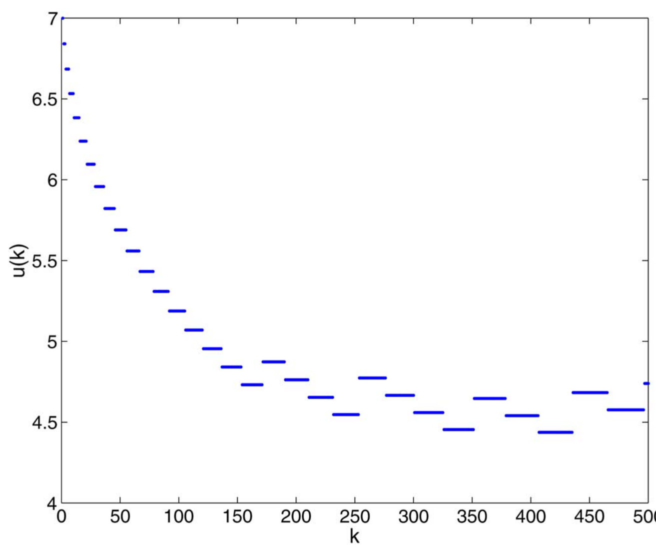

Fig. 4. Adaptive control law.

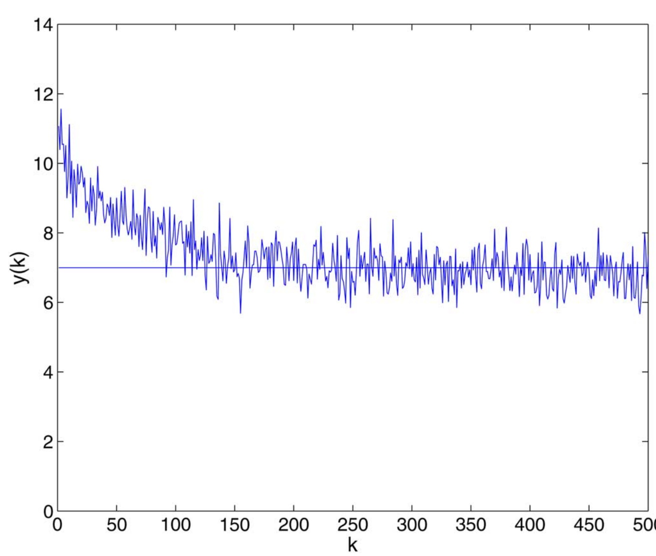

Fig. 5. System output with tracking target $y^* = 7$.

VI. Extended Works

This technical note just shows the idea of two-scale adaptive control algorithm with binary-valued observations. There are many extended works that can be considered towards the directions of system models, noise properties and the multiple set-valued case.

System models: The empirical measure method works for far beyond the FIR model with set-valued observations. The methodology for the IIR (Infinite Impulse Response) model can be found in [18], and even for nonlinear models such as Wiener model ([19]) and Hammerstein model ([17]).

Noise properties: Noise properties are very important for empirical measure method. This technical note considers the case where the noise distribution is known and other constrains required in Assumption 2.1. It should be pointed out that this methodology still works for the case of unknown distributions ([18]) and more general noises such as mixing noises ([20]).

Set-valued observation: The output observations are binary in this technical note, which contain the information with only one threshold. For the multiple threshold cases, the estimation algorithm can be constructed as a quasi-convex combination of information from each thresholds ([17]). It can be shown that with the increasing of the threshold, the convergence speed can be faster.

VII. Conclusion

This technical note studies the adaptive control for linear systems with set-valued observations to track periodic targets. For the different order of system parameters, accessorial parameters with the same order as the one of tracking target can be constructed. A two-scale adaptive control algorithm is constructed. The k-th control signal lasts for a holding time k, during which the accessorial parameter estimate is constructed and shown to be convergent and asymptotically efficient. As the holding time goes to infinity, the adaptive tracking algorithm is proved to be asymptotically optimal and the convergence rate is also obtained.

For different cases of system parameter η with order n and target vector $\mathbf{y}^*$ with order m, the adaptive tracking problem can be written and turn to the relationship

$$\Phi\Theta = Y$$

by (8). For $n \le m$, all system parameters can be estimated. However, as $n > m$, only m constructed parameters b_i $(i = 1, \ldots, m)$ can be estimated instead of the whole system parameters. Even though, it is interesting to find that the estimates of constructed parameters are good enough to track the target. As a result, for all cases of n and m, the tracking purpose is achieved.

The holding time is a specialty of two-scale adaptive algorithm and the constrain in this technical note is that the holding time goes to infinity. Meaningful future works contain that constructing algorithms with finite holding time, and even the identification and control algorithms are constructed at each time step just as the classic cases.

Appendix A
Proof of Proposition 1

To prove Proposition 1, the following lemma is introduced first.

Lemma A.1: ([16]): The circulant matrix $\mathbf{T}([b_m, \ldots, b_1])$ is of full rank if and only if its discrete Fourier transform $\gamma_k = \sum_{i=1}^{m} b_i e^{-i\omega_k j}$ is nonzero at $\omega_k = 2\pi k/m$ with $k = 1, \ldots, m$, where j is the imaginary unit, i.e, $j^2 = -1$.

Now, we can give the proof of Proposition 1. By Lemma A.1, we only need to confirm that γ_k is nonzero at $\omega_k = 2\pi k/m$ with $k = 1, \ldots, m$. Assumption 2.3 ensures the fact that for $|z| \le 1$ we have $B(z) = a_1 + a_2 z + \cdots + a_n z^{n-1} \neq 0$.

As $n \le m$, we have $b_i = 0$ for $i > n$, and thus

$$\gamma_k = \sum_{i=1}^{m} b_i e^{-i\omega_k j} = \sum_{i=1}^{n} b_i e^{-i\omega_k j} = \sum_{i=1}^{n} a_i e^{-i\omega_k j}.$$

Since $B(z) \neq 0$ for $|z| \le 1$, together with the fact that the points ω_k follows $|\omega_k| = 1$, we have $\sum_{j=1}^{m} a_j e^{-i\omega_k j}$ is nonzero at ω_k. Hence, $\mathbf{T}([b_m, \ldots, b_1])$ is of full rank.

As $n > m$, for $\omega_k = 2\pi k/m$ with $k = 1, \ldots, m$, by $B(z) \neq 0$ for $|z| \le 1$ and $i = 1, \ldots, m$, we have

$$b_i e^{-i\omega_k j} = \sum_{l=0}^{\left\lfloor \frac{(n-i)}{m} \right\rfloor} a_{i+lm} e^{-i\omega_k j} = \sum_{l=0}^{\left\lfloor \frac{(n-i)}{m} \right\rfloor} a_{i+lm} e^{-(i+lm)\omega_k j}$$

and $\sum_{i=1}^{m} b_i e^{-i\omega_k j} = \sum_{i=1}^{n} a_j e^{-i\omega_k j} \neq 0$. Hence, $\mathbf{T}([b_m, \ldots, b_1])$ is of full rank.

Thus, the proposition is true. ∎

References

[1] H. F. Chen and L. Guo, *Identification and Stochastic Adaptive Control*. Boston, MA: Birkhäuser, 1991.

[2] G. C. Goodwin and K. S. Sin, *Adaptive Filtering, Prediction and Control*. Englewood Cliffs, N.J.: Prentice-Hall, 1984.

[3] T. Bohlin, "On the maximum likelihood method of identification," *IBM J. Res. Develop.*, vol. 14, no. 1, pp. 41–51, 1970.

[4] V. Verdult, N. Bergboer, and M. verhaegen, "Maximum likelihood identification of multivariable bilinear state-space systems by projected gradient search," in *Proc. 41st IEEE Conf. Decision Control*, Las Vegas, NV, Dec. 2002, pp. 1808–1813.

[5] R. E. Kalman, "A new approach to linear filtering and prediction problems," *Trans. ASME, Series D, J. Basic Eng.*, vol. 82, pp. 35–45, Mar. 2002.

[6] L. Guo, "Further results on least squares based minimum variance adaptive control," *SIAM J. Control Optim.*, vol. 32, no. 1, pp. 187–212, 1994.

[7] L. Zhang and B. Zhang, "A geometrical representation of McCulloch-Pitts neural model and its applications," *IEEE Trans. Neural Networks*, vol. 10, no. 4, pp. 925–929, Apr. 1999.

[8] I. Akyildiz, W. Su, Y. Sankarasubramaniam, and E. Cayirci, "Wireless sensor networks: A survey," *Computer Networks*, vol. 38, pp. 393–422, 2002.

[9] K. You, L. Xie, S. Sun, and W. Xiao, "Multiple-level quantized innovation Kalman filter," in *Proc. 17th IFAC World Congress*, Jul. 2008, pp. 1420–1425.

[10] J. C. Agüero, G. C. Goodwin, and J. I. Yuz, "System identification using quantized data," in *Proc. 46th IEEE Conf. Decision Control*, 2007, pp. 4263–4268.

[11] L. Y. Wang, C. Y. Li, G. Yin, L. Guo, and C. Z. Xu, "State observability and observers of linear-time-invariant systems under irregular sampling and sensor limittations," *IEEE Trans. Autom. Control*, vol. 56, no. 11, pp. 2639–2654, Nov. 2011.

[12] L. Y. Wang, J. F. Zhang, and G. G. Yin, "System identification using binary sensors," *IEEE Trans. Autom. Control*, vol. 48, no. 11, pp. 1892–1907, Nov. 2003.

[13] L. Y. Wang and G. Yin, "Asymptotically efficient parameter estimation using quantized output observations," *Automatica*, vol. 43, pp. 1178–1191, 2007.

[14] Y. L. Zhao and J. Guo, "Quantization system identification under a class of persistent excitations," in *Proc. 29th Chinese Control Conf.*, Beijng, China, Jul. 29–31, 2010, pp. 6221–6226.

[15] J. Guo, J. F. Zhang, and Y. L. Zhao, "Adaptive tracking control of a class of first-order systems with binary-valued observations and time-varying thresholds," *IEEE Trans. Autom. Control*, vol. 56, no. 11, pp. 2991–2996, Nov. 2011.

[16] P. Lancaster and M. Tismenetsky, *The Theory of Matrices*, 2nd ed. New York: Academic Press, 1985.

[17] Y. L. Zhao, J. F. Zhang, L. Y. Wang, and G. G. Yin, "Identification of Hammerstein systems with quantized observations," *SIAM J. Control Optim.*, vol. 48, no. 7, pp. 4352–4376, 2010.

[18] L. Y. Wang, G. G. Yin, and J. F. Zhang, "Joint identification of plant rational models and noise distribution functions using binary-valued observations," *Automatica*, vol. 42, no. 4, pp. 535–547, 2006.

[19] Y. L. Zhao, L. Y. Wang, G. G. Yin, and J. F. Zhang, "Identification of Wiener systems with binary-valued output observations," *Automatica*, vol. 43, no. 10, pp. 1752–1765, 2007.

[20] G. Yin, S. Kan, L. Y. Wang, and C. Z. Xu, "Identification of systems with regime switching and unmodelled dynamics," *IEEE Trans. Autom. Control*, vol. 54, no. 1, pp. 34–47, Jan. 2009.